

# Response to Shafer's Comments

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I'd like to thank Professor Shafer for the historical background and additional perspective he has provided. I have only a few comments in response. I will proceed from less to more substantive issues.

## 1 Use of the word "plausibility"

Professor Shafer writes,

When participants in a debate appropriate the other side's terms of discourse in a way that contradicts the dictionary, the coherence and civility of the debate is imperiled.

I have to admit to being mystified as to what contradiction Shafer sees between his dictionary definition of "plausibility," and my use of that term, as I can find none. More to the point, the charge of appropriating terminology could with more justice be leveled against the theory of belief functions, as Jaynes' use of "plausibility" in the context of Cox's Theorem goes back at least as far as 1958 [1], which predates Dempster's seminal work on belief functions [2] by ten years, and Shafer's work [3,4] by fifteen years.

From a pragmatic standpoint, though, I concur that it would be worthwhile to avoid using the term "plausibility" in future discussions of Cox's Theorem, due to that term's prominence in the theory of belief functions. The suggested replacement, "likeliness," is too closely tied to the notion of probability: my dictionary explicitly defines it as the probability of an outcome. Looking for alternatives, I found the term "credible" defined as "capable of being credited or believed; worthy of belief," and credibility defined as "the quality, capability, or power to elicit belief." Even here we run into a possible conflict with the

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use of “credibility” by Smets [5]; however, his usage is also in the context of deriving a system for uncertain reasoning from axioms. As that is compatible with the goals of this work, I shall use the term “credibility” in place of “plausibility” in the rest of this note.

## 2 Bernstein’s Work

Bernstein’s approach to deriving the rules of probability theory is an interesting alternative to Cox’s approach. It appears that Bernstein’s axioms suffice to give us the product rule for all domains and propositions, assuming that the axioms include some analog of Cox’s implicit universality axiom. The sum rule  $p(A | X) + p(\neg A | X) = 1$  appears problematic, however: it depends on writing  $A$  as  $\bigvee_{i=1}^m B_i$ , where the  $B_i$  are  $n$  equally likely, mutually exclusive and exhaustive propositions, and observing that  $p(A | X) = m/n$  whereas  $p(\neg A | X) = (n - m)/n$ . It is not clear from Shafer’s description that Bernstein’s axioms force the sum rule to also hold true for arbitrary propositions  $A$  that may not be expressible in this form. This full generality is achieved, however, if we add two Cox axioms: universality and the requirement that the credibility of  $\neg A$  be a continuous function of the credibility of  $A$ .

Interesting and revealing though Bernstein’s work may be, it is ultimately irrelevant to judging the appropriateness of Cox’s axioms. These must be judged on their own merits, and not by conformance to a preferred approach for deriving probability theory. (After all, the point of the exercise is not to derive probability theory specifically, but a logic of uncertain reasoning.) In particular, Shafer criticizes Cox for not following Bernstein in using the concept of equally likely cases, and opines that this makes Cox’s reasoning unpersuasive. Yet Shafer believes that “the concept of equally likely cases provides one way of seeing” why the probability calculus is “a special, not universal, framework for uncertain reasoning.” If so, why criticize an effort to construct a *universal* framework for uncertain reasoning for failing to employ this concept?

## 3 The Value of Simplicity

As Shafer notes, I do not claim that the modified Cox axioms I present are completely compelling; however, they seem to be the *simplest* requirements one could impose while still maintaining compatibility with the propositional calculus and commonsense reasoning. The propositional calculus is compositional: the truth value of a compound proposition is a function only of the

truth values of its component sub-propositions and the logical operator applied. This property gives us a simple recursive procedure for evaluating the truth value of a compound expression, that terminates when we have decomposed the problem down to primitive propositions. In generalizing the propositional calculus to deal with uncertainty, this compositionality is a desirable property to retain.

Consider R3:  $(\neg A | X) = S(A | X)$  for some nonincreasing function  $S$ . This is a direct analog of the rule for negation in the propositional calculus, and the simplest choice we could make. It is such a natural choice, in fact, that both possibility theory and D-S belief-function theory arguably satisfy R3!

In possibility theory [6] one may view the credibility of a proposition  $A$  as having two coordinates: the possibility of  $A$  and necessity of  $A$ . In symbols,

$$(A | X) = (\Pi_X(A), N_X(A)),$$

where  $N_X(A) = 1 - \Pi_X(\neg A)$ . Then  $(\neg A | X) = S(A | X)$ , where  $S(x, y) = (1 - y, 1 - x)$ . The natural ordering for these credibilities is a partial order, with credibility increasing whenever possibility or necessity increases without a decrease in the other; that is,  $(x, y) \leq (x', y')$  iff  $x \leq x'$  and  $y \leq y'$ . Under this ordering  $S$  is a strictly decreasing function.

In D-S belief-function theory [4] the credibility of a proposition  $A$  again may be viewed as having two coordinates: the degree of belief in  $A$  and the degree of doubt in  $A$ . In symbols,

$$(A | X) = (\text{Bel}_X(A), \text{Dou}_X(A)),$$

where  $\text{Dou}_X(A) = \text{Bel}_X(\neg A)$ . Then  $(\neg A | X) = S(A | X)$ , where  $S(x, y) = (y, x)$ . The natural ordering for these credibilities is a partial order, with credibility increasing when belief increases without doubt increasing, or when doubt decreases without belief decreasing; that is,  $(x, y) \leq (x', y')$  iff  $x \leq x'$  and  $y \geq y'$ . Under this ordering  $S$  is again a strictly decreasing function.

Likewise, we would like to have  $(A \wedge B | X) = F((A | X), (B | X))$ , in direct analogy to the propositional calculus and as the simplest choice we could make; but this choice is too simple, having unacceptable consequences, as I have already discussed. The simplest choice that retains some form of compositionality is  $(A \wedge B | X) = F((A | B, X), (B | X))$  for some function  $F$ . Shafer asks why we should always use the same function  $F$ ; my answer is that this is the simplest approach. Furthermore, it is closest in spirit to the propositional calculus, in which we always use the same function to combine the truth values of  $A$  and  $B$  in evaluating the truth of the conjunction  $A \wedge B$ .

## 4 Conclusion

Although there is not a completely compelling case for Cox's axioms, and thus one cannot claim that probability theory is the only workable logic of uncertain reasoning, there are strong grounds for a weaker, but still interesting, claim: probability theory is the *simplest* workable logic of uncertain reasoning one could hope to construct. I suspect that Professor Shafer might be willing to agree with me on this point, although he would argue that its simplicity limits its generality, while I would question the need for the added complexity of belief-function theories. Ultimately, this matter can only be settled by extended experience applying these theories to real problems, and so I do not expect to see a resolution for many decades.

## References

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