Partial Solution to Exercise 4.1 of PTLOS

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This solution for the case $m = 2$ (two data points) and $n = 3$ (three hypotheses) is due to Timothy D. Sanders; some reformatting and minor modifications have been made to his original solution.

For later convenience, we shall define

$$\begin{aligned} h_i &= P(H_i) \\ d_{ji} &= P(D_j \mid H_i). \end{aligned}$$

We assume that $h_2, h_3 > 0$, as otherwise we effectively have $n < 3$. We use $\neg$ for NOT, $\vee$ for OR, and $\wedge$ for AND.

Since the hypotheses H_i are exhaustive, $\neg H_i$ is equivalent to the OR of all hypotheses but i, that is, $\bigvee_{k \neq i} H_k$. Thus,

$$\begin{aligned} P(D_j \mid \neg H_i) &= P\left(D_j \mid \bigvee_{k \neq i} H_k\right) \\ &= \frac{P(D_j \wedge (\bigvee_{k \neq i} H_k))}{P(\bigvee_{k \neq i} H_k)} \\ &= \frac{\sum_{k \neq i} P(D_j \wedge H_k)}{\sum_{k \neq i} P(H_k)} \\ &= \frac{\sum_{k \neq i} d_{jk} h_k}{\sum_{k \neq i} h_k}. \end{aligned} \tag{1}$$

(The last two equalities use the fact that the H_k are mutually exclusive hypotheses.) In other words, $P(D_j \mid \neg H_i)$ is a weighted average of $P(D_j \mid H_k)$, $k \neq i$.

From the product rule and (Jaynes, 4.28) we have

$$\begin{aligned} P\left(\bigwedge_j D_j \wedge H_i\right) &= P(H_i)\, P\left(\bigwedge_j D_j \mid H_i\right) \\ &= h_i \prod_j d_{ji}. \end{aligned} \tag{2}$$

From (Jaynes, 4.29) we obtain

$$\begin{aligned}
\prod_j P(D_j \mid \neg H_i) &= P\left(\bigwedge_j D_j \mid \neg H_i\right) \\
&= P\left(\bigwedge_j D_j \mid \bigvee_{k\neq i} H_k\right) \\
&= \frac{P(\bigwedge_j D_j \wedge (\bigvee_{k\neq i} H_k))}{P(\bigvee_{k\neq i} H_k)} \\
&= \frac{\sum_{k\neq i} P(\bigwedge_j D_j \wedge H_k)}{\sum_{k\neq i} h_k}. \qquad (3)
\end{aligned}$$

Combining (2) and (3) we then obtain

$$\prod_j P(D_j \mid \neg H_i) \cdot \sum_{k\neq i} h_k = \sum_{k\neq i} h_k \prod_j d_{jk}. \qquad (4)$$

Next use (1) to eliminate $P(D_j \mid \neg H_i)$ from the left-hand side of (4):

$$\prod_j \sum_{k\neq i} d_{jk} h_k = \left(\sum_{k\neq i} h_k\right)^{m-1} \sum_{k\neq i} h_k \prod_j d_{jk}. \qquad (5)$$

In the specific case of $m = 2$ and $n = 3$, for $i = 1$ this gives

$$(d_{12}h_2 + d_{13}h_3)(d_{22}h_2 + d_{23}h_3) = (h_2 + h_3)(d_{12}d_{22}h_2 + d_{13}d_{23}h_3). \qquad (6)$$

Expanding both sides of (6) as polynomials in h_2 and h_3, we find that the h_2^2 and h_3^2 terms cancel out, leaving

$$(d_{13}d_{22} + d_{12}d_{23})h_2h_3 = (d_{12}d_{22} + d_{13}d_{23})h_2h_3;$$

since $h_2, h_3 > 0$, this gives

$$d_{13}d_{22} + d_{12}d_{23} = d_{12}d_{22} + d_{13}d_{23},$$

or, rearranging terms,

$$(d_{13} - d_{12})(d_{22} - d_{23}) = 0. \qquad (7)$$

Likewise, for $i = 2$ and $i = 3$ we obtain

$$\begin{aligned}
(d_{13} - d_{11})(d_{21} - d_{23}) &= 0 \\
(d_{11} - d_{12})(d_{22} - d_{21}) &= 0. \qquad (8)
\end{aligned}$$

Thus we have

$$\begin{aligned} &(d_{13} = d_{12} \text{ or } d_{22} = d_{23}) \\ \text{and} \quad &(d_{13} = d_{11} \text{ or } d_{21} = d_{23}) \\ \text{and} \quad &(d_{11} = d_{12} \text{ or } d_{22} = d_{21}). \end{aligned}$$

This gives a total of $2^3 = 8$ possibilities; each of these 8 possibilities yields either $d_{11} = d_{12} = d_{13}$ or $d_{21} = d_{22} = d_{23}$. But $d_{j1} = d_{j2} = d_{j3}$, when combined with (1), yields

$$P(D_j \mid \neg H_i) = \frac{\sum_{k \neq i} d_{jk} h_k}{\sum_{k \neq i} h_k} = d_{ji} = P(D_j \mid H_i).$$

Hence only D_1 or D_2, but not both, can produce any updating of the probabilities for H_i.